4.3 - Homogeneous Linear Equations with Constant Coefficients

Consider the differential equation $y'' - 5y' + 6y = 0$.

Start with $y = e^{mx} \Rightarrow y' = me^{mx}, y'' = m^2 e^{mx}$

$m^2 \cancel{e^{mx}} - 5m\cancel{e^{mx}} + 6\cancel{e^{mx}} = 0$

Auxiliary equation

$m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0$

$m = 2, 3$

$y_1 = e^{2x}, \quad y_2 = e^{3x}$

General solution: $y = c_1 e^{2x} + c_2 e^{3x}$

Definition: The **auxiliary equation** of the second-order linear differential equation $ay'' + by' + cy = 0$ is $am^2 + bm + c = 0$.

The auxiliary equation has solutions

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This leads to 3 cases:

Case I: $m_1 \neq m_2$, both are real.

Then $\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

Case II: $m_1 = m_2$, real. Let's use $m = m_1$

$b^2 - 4ac = 0 \Rightarrow m = -\frac{b}{2a}$

$y_1 = e^{-b/2a\,x}$, $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$

Reduction of order: $y_2 = e^{-b/2a\,x} \displaystyle\int \frac{e^{-\int \frac{b}{a}dx}}{e^{-b/a\,x}}\,dx$

$y_2 = e^{mx}(x)$

General solution: $\boxed{y = c_1 e^{mx} + c_2 x e^{mx}}$

Case III: $m = \alpha \pm \beta i$

$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$

$y = e^{\alpha x}\left(c_1 e^{\beta i x} + c_2 e^{-\beta i x}\right)$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$-e^{-i\theta} = -\cos\theta + i\sin\theta$

$\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$

$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta$

Strategic choices for c_1, c_2 give us

If $m = \underbrace{\alpha}_{\text{Real}} \pm \beta i$, the general solution is

Imaginary

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

So if $m = 3 \pm 5i$, we get

$$y = e^{3x}(c_1 \cos 5x + c_2 \sin 5x)$$

Example: Find the general solution of the given second-order differential equation.

$y'' - 10y' + 25y = 0$

Aux eqn: $m^2 - 10m + 25 = 0$

$(m-5)^2 = 0 \Rightarrow m = 5$, repeated

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

Example: $2y'' - 3y' + 4y = 0$

$2m^2 - 3m + 4 = 0$

$$m = \frac{3 \pm \sqrt{9-32}}{4} = \frac{3}{4} \pm \frac{\sqrt{23}}{4} i$$

$$\boxed{y = e^{3/4 x}\left(c_1 \cos \frac{\sqrt{23}}{4} x + c_2 \sin \frac{\sqrt{23}}{4} x\right)}$$

Example: Find the general solution of the given higher-order differential equation.

$$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4x = 0$$

$$m^3 - m^2 - 4 = 0$$

wolframalpha.com : $m = 2, \; -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$

$$x = c_1 e^{2t} + e^{-1/2 x}\left(c_2 \cos\frac{\sqrt{7}}{2}x + c_3 \sin\frac{\sqrt{7}}{2}x\right)$$

Special 2nd-order differential equations:
$y'' + k^2 y = 0$ and $y'' - k^2 y = 0$, $k \in \mathbb{R}$

$m^2 + k^2 = 0$

$m = \pm ki$

$y = c_1 \cos kx + c_2 \sin kx$

$m = \pm k$

$y = c_1 e^{kx} + c_2 e^{-kx}$

can become

$y = c_1 \cosh kx + c_2 \sinh kx$

Recall $\cosh x = \frac{e^x + e^{-x}}{2}$ ($\sinh$, $-$)

Example: Find a homogeneous linear differential equation with constant coefficients whose general solution is given.

$y = c_1 e^{-4x} + c_2 e^{-3x}$

Example: $y = c_1 e^{10x} + c_2 x e^{10x}$

Example: $y = c_1 + c_2 e^{2x} \cos 5x + c_3 e^{2x} \sin 5x$